Extra Lecture: The Set IBr(G) — Irreducible Brauer Characters

Goal: To rigorously define and explore the set $\mathrm{IBr}(G)$, the irreducible Brauer characters of a finite group G over a field of positive characteristic p, and to understand their role in modular representation theory alongside $\mathrm{Irr}(G)$, decomposition matrices, and block theory.

1. Modular Setting and Motivation

Let:

- G be a finite group,
- F a field of characteristic $p > 0$,
- $\overline{F}$ an algebraic closure of F.

In modular representation theory, we study $\overline{F}[G]$-modules. Since $p \mid |G|$, the group algebra $\overline{F}[G]$ is *not semisimple*, so representations may not decompose into irreducibles in the usual way.

2. Definitions

Definition E.1. A *Brauer character* is a class function $\varphi : G_{p'} \to \overline{F}$, where $G_{p'}$ is the set of p-regular elements (those whose order is not divisible by p), associated to a representation $\rho : G \to \mathrm{GL}_n(\overline{F})$.

Definition E.2. The set $\mathrm{IBr}(G)$ is the set of irreducible Brauer characters of G over $\overline{F}$. These correspond to isomorphism classes of simple $\overline{F}[G]$-modules.

Definition E.3. A class function $\varphi : G_{p'} \to \overline{F}$ is a Brauer character if it is the trace function of an absolutely irreducible $\overline{F}[G]$-representation.

3. Properties of $\mathrm{IBr}(G)$

Theorem E.4. The number of elements in $\mathrm{IBr}(G)$ equals the number of isomorphism classes of simple $\overline{F}[G]$-modules.

Theorem E.5 (Orthogonality). The Brauer characters form an orthonormal basis of the space of class functions on $G_{p'}$, under the modified inner product:

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G_{p'}} |C_G(g)| \varphi(g) \overline{\psi(g)}.$$

4. Relation to $\mathrm{Irr}(G)$ and Decomposition Matrices

Theorem E.6. Each $\chi \in \mathrm{Irr}(G)$ reduces mod p to a (possibly reducible) Brauer character:

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi} \varphi.$$

This gives rise to the *decomposition matrix* $D = (d_{\chi\varphi})$, which records how ordinary irreducibles reduce modulo p in terms of irreducible Brauer characters.

- Rows of D: indexed by $\mathrm{Irr}(G)$,
- Columns of D: indexed by $\mathrm{IBr}(G)$,
- Entries: $d_{\chi\varphi} \in \mathbb{Z}_{\geq 0}$.

5. Example

Example E.7 (S_3 mod 2):

- $|S_3| = 6$, $p = 2$,
- $\mathrm{Irr}(S_3) : \chi_1, \chi_2, \chi_3$,
- $\mathrm{IBr}(S_3) : \varphi_1, \varphi_2$,
- Decomposition matrix:

χ	φ_1	φ_2
χ_1	1	0
χ_2	0	1
χ_3	1	1

Observation: χ_3 (degree 2) reduces to the sum of the two Brauer characters.

6. Counterexamples and Pitfalls

- $\mathrm{IBr}(G) \nsubseteq \mathrm{Irr}(G)$: they are characters in a different characteristic.
- A single $\chi \in \mathrm{Irr}(G)$ may decompose into several $\varphi \in \mathrm{IBr}(G)$.
- The number of Brauer characters is not the number of conjugacy classes, but the number of p-regular conjugacy classes.

7. Summary

In this lecture, we have:

- Defined the set $\mathrm{IBr}(G)$ and its role in modular representation theory,
- Understood the connection between ordinary characters and Brauer characters via decomposition matrices,
- Explored orthogonality of Brauer characters and their domain restriction to p-regular elements,
- Seen that $\mathrm{IBr}(G)$ encapsulates the simple module structure over $\overline{\mathbb{F}}_p$.